Clique-width of Restricted Graph Classes

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Motivation

Most natural problems in algorithmic graph theory are NP-complete.

Want to find restricted classes of graphs where we can solve some problems in polynomial time.

Best if we can find classes where lots of problems can be solved in polynomial time.

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Why Clique-width?

Theorem (Courcelle, Makowsky and Rotics 2000, Kobler and Rotics 2003, Rao 2007, Oum 2008, Grohe and Schweitzer 2015)

Any problem expressible in "monadic second-order logic with quantification over vertices" (and certain other classes of problems) can be solved in polynomial time on graphs of bounded clique-width.

This includes:

- **Vertex Colouring**
- Maximum Independent Set
- Minimum Dominating Set
- Hamilton Path/Cycle
- Partitioning into Perfect Graphs
- Graph Isomorphism
- ...

Clique-width

The clique-width is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i (represented by $i(v)$)

(ii) taking the disjoint union of two labelled graphs G_1 and G_2 (represented by $G_1 \oplus G_2$)

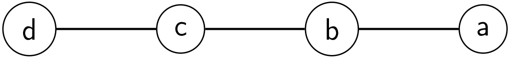
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$) (represented by $\eta_{i,j}$)

(iv) renaming label i to j (represented by $\rho_{i \to j}$)

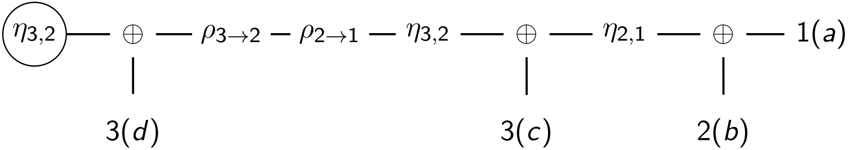
For example, P_4 has clique-width 3.

An expression for a graph can be represented by a rooted tree.

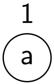
Clique-width



$$\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))$$



Clique-width



$1(a)$

$1(a)$

Clique-width

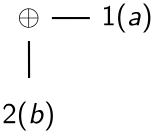
$2(b)$ $1(a)$

$1(a)$

$2(b)$

Clique-width

$$2(b) \oplus 1(a)$$



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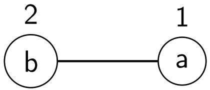
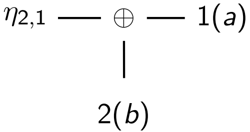


$\textcolor{red}{\eta_{2,1}}(2(b) \oplus 1(a))$



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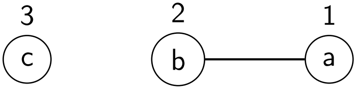
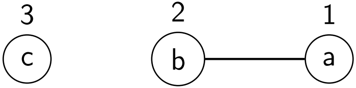


$3(c)$ $\quad \eta_{2,1}(2(b) \oplus 1(a))$

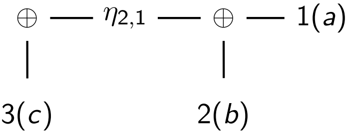
$\eta_{2,1}$ — $\oplus$ — $1(a)$

$3(c)$ $\quad 2(b)$

Clique-width



$$3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))$$



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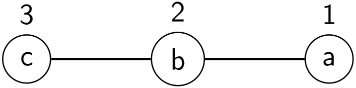

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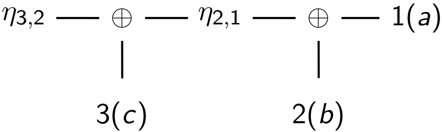

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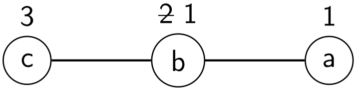
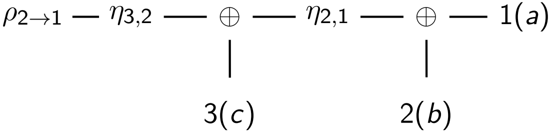


$$\color{red}{\rho_{2\to 1}}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))$$



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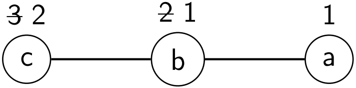
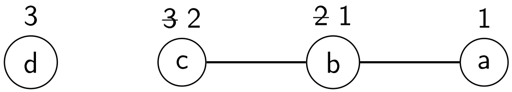


$$\color{red}{\rho_{3\to 2}}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))$$

$$\rho_{3\to 2} - \rho_{2\to 1} - \eta_{3,2} - \oplus - \eta_{2,1} - \oplus - 1(a)$$

$$3(c) \qquad 2(b)$$

Clique-width



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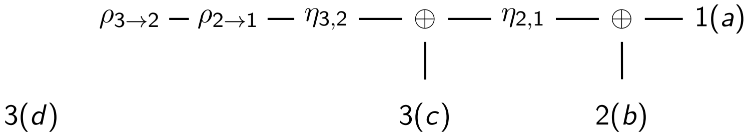

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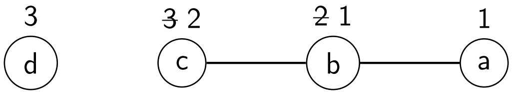

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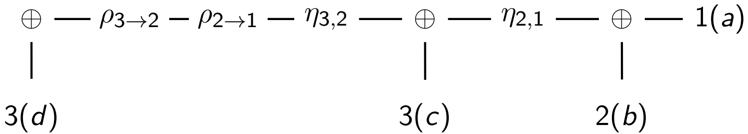

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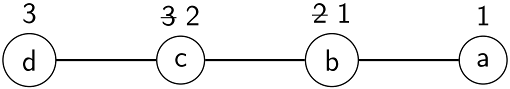

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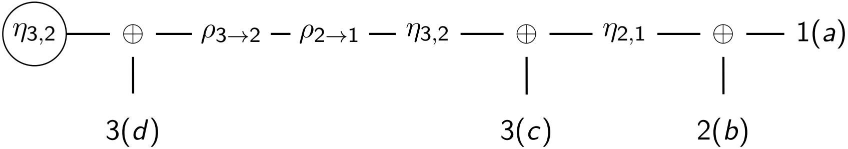

Calculating clique-width

Theorem (Fellows, Rosamond, Rotics, Szeider 2009)

Calculating clique-width is NP-hard.

Theorem (Corneil, Habib, Lanlignel, Reed, Rotics 2012)

Can detect graphs of clique-width at most 3 *in polynomial time.*

It's not known if this is also the case for graphs of clique-width 4.

Theorem (Oum 2008)

Can find a c-expression for a graph G where $c \leq 8^{\mathrm{cw}(G)} - 1$ in cubic time.

The clique-width of all graphs up to 10 vertices has been calculated (Heule & Szeider 2013).

Why clique-width?

- "Equivalent" to rank-width and NLC-width
- Generalises tree-width
- "Equivalent" to tree-width on graphs of bounded degree

The following operations don't change the clique-width by "too much"

- Complementation
- Bipartite complementation
- Vertex deletion
- Edge subdivision (for graphs of bounded-degree)

Need only look at graphs that are

- prime
- 2-connected

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Underlying Research Question

What kinds of graph properties ensure bounded clique-width?

By knowing what the bounded cases are, we may be able to reduce other classes down to known cases and get polynomial algorithms.

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What kinds of graph properties ensure bounded clique-width?

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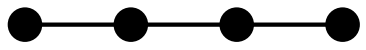
Hereditary Classes

A graph H is an induced subgraph of G if H can be obtained by deleting vertices of G, written $H \subseteq_i G$.



P_4

$3P_1$

$P_1 + P_2$

So $P_1 + P_2 \subseteq_i P_4$, but $3P_1 \not\subseteq_i P_4$.

A class of graphs is hereditary if it is closed under taking induced subgraphs.

Let S be a set of graphs. The class of S-free graphs is the set of graphs that do not contain any graph in S as an induced subgraph.

For example: bipartite graphs are the $(C_3, C_5, C_7, \ldots)$-free graphs

We will consider classes defined by finite set of forbidden induced subgraphs.

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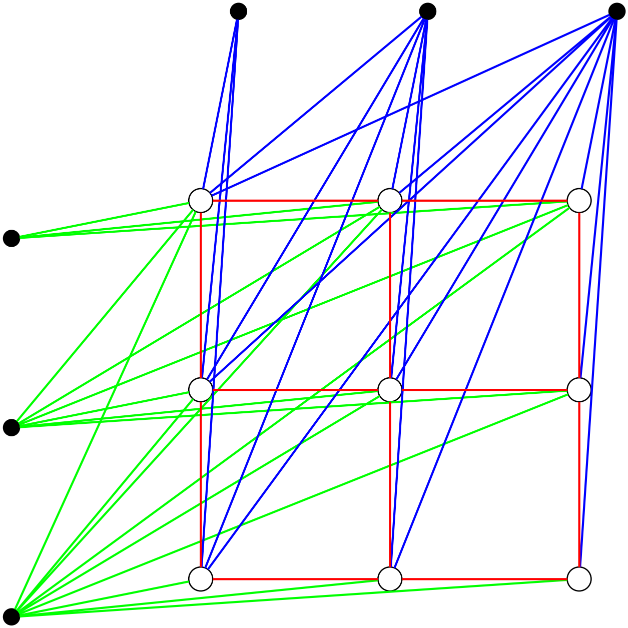
Graphs of large clique-width

Theorem (General Construction)

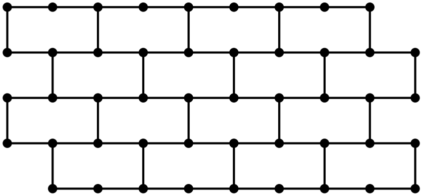
For $m \geq 0$ and $n > m+1$ the clique-width of a graph G is at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ if $V(G)$ has a partition into sets $V_{i,j} (i, j \in \{0, \ldots, n\})$ with the following properties:

- $|V_{i,0}| \leq 1$ *for all* $i \geq 1$.
- $|V_{0,j}| \leq 1$ *for all* $j \geq 1$.
- $|V_{i,j}| \geq 1$ *for all* $i, j \geq 1$.
- $G[\cup_{j=0}^{n} V_{i,j}]$ *is connected for all* $i \geq 1$.
- $G[\cup_{i=0}^{n} V_{i,j}]$ *is connected for all* $j \geq 1$.
- *For* $i, j, k \geq 1$, *if a vertex of* $V_{k,0}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $i \leq k$.
- *For* $i, j, k \geq 1$, *if a vertex of* $V_{0,k}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $j \leq k$.
- *For* $i, j, k, \ell \geq 1$, *if a vertex of* $V_{i,j}$ *is adjacent to a vertex of* $V_{k,\ell}$ *then* $|k - i| \leq m$ *and* $|\ell - j| \leq m$.



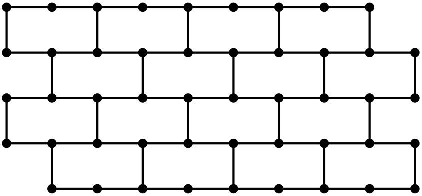
Graphs of large clique-width

Example:



Walls are bipartite and have unbounded clique-width, even if we subdivide each edge k times.

Graphs of large clique-width



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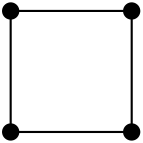
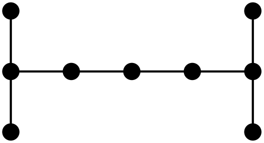


C_4



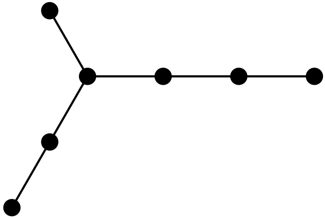
I_4

If H contains a C_k or I_k, then the k-subdivided walls are H-free.

Which classes have bounded clique-width?

If the class of H-free graphs has bounded clique-width then H must contain no cycles and no I_k.

Every component of H must be a subdivided claw, path or isolated vertex. The set of such graphs is called $\mathcal{S}$.



$S_{1,2,3}$

P_5

P_1

H-free graphs

Theorem (D., Paulusma 2015)

The class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$.

Colouring H-free graphs

Theorem (Král', Kratochvíl, Tuza & Woeginger, 2001)

The Vertex Colouring *problem is polynomial-time solvable for H-free graphs if and only if $H \subseteq_i P_1 + P_3$ or P_4, otherwise it is NP-complete.*

$P_1 + P_3$

P_4

Colouring (H_1, H_2)-free graphs

The Vertex Colouring problem is polynomial-time solvable for (H_1, H_2)-free graphs if

1. H_1 or H_2 is an induced subgraph of $P_1 + P_3$ or of P_4
2. $H_1 \subseteq_i K_{1,3}$, and $H_2 \subseteq_i C_3^{++}$, $H_2 \subseteq_i C_3^*$ or $H_2 \subseteq_i P_5$
3. $H_1 \neq K_{1,5}$ is a forest on at most six vertices or $H_1 = K_{1,3} + 3P_1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
4. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 = K_t$ for $t \geq 4$
5. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
6. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_1 + P_4}$
7. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_5}$
8. $H_1 \subseteq_i 2P_1 + P_2$, and $H_2 \subseteq_i \overline{2P_1 + P_3}$ or $H_2 \subseteq_i \overline{P_2 + P_3}$
9. $H_1 \subseteq_i \overline{2P_1 + P_2}$, and $H_2 \subseteq_i 2P_1 + P_3$ or $H_2 \subseteq_i P_2 + P_3$
10. $H_1 \subseteq_i sP_1 + P_2$ for $s \geq 0$ or $H_1 = P_5$, and $H_2 \subseteq_i \overline{tP_1 + P_2}$ for $t \geq 0$
11. $H_1 \subseteq_i 4P_1$ and $H_2 \subseteq_i \overline{2P_1 + P_3}$
12. $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i C_4$ or $H_2 \subseteq_i \overline{2P_1 + P_3}$.

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4. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 = K_t$ for $t \geq 4$
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6. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_1 + P_4}$
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8. $H_1 \subseteq_i 2P_1 + P_2$, and $H_2 \subseteq_i \overline{2P_1 + P_3}$ or $H_2 \subseteq_i \overline{P_2 + P_3}$
9. $H_1 \subseteq_i \overline{2P_1 + P_2}$, and $H_2 \subseteq_i 2P_1 + P_3$ or $H_2 \subseteq_i P_2 + P_3$
10. $H_1 \subseteq_i sP_1 + P_2$ for $s \geq 0$ or $H_1 = P_5$, and $H_2 \subseteq_i \overline{tP_1 + P_2}$ for $t \geq 0$
11. $H_1 \subseteq_i 4P_1$ and $H_2 \subseteq_i \overline{2P_1 + P_3}$
12. $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i C_4$ or $H_2 \subseteq_i \overline{2P_1 + P_3}$.

The class of (H_1, H_2)-free graphs has bounded clique-width if:

1. H_1 or $H_2 \subseteq_i P_4$;
2. $H_1 = sP_1$ and $H_2 = K_t$ for some s, t;
3. $H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1$, $K_{1,3} + P_2$, $P_1 + S_{1,1,2}$, P_6 or $S_{1,1,3}$;
4. $H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$, $3P_1 + P_2$ or $P_2 + P_3$;
5. $H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or P_5;
6. $H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;
7. $H_1, \overline{H_2} \subseteq_i K_{1,3}$.

and it has unbounded clique-width if:

1. $H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;
2. $\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;
3. $H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;
4. $H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3}$, $5P_1$, $P_2 + P_4$ or P_6;
5. $H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2$, $2P_1 + P_4$, $4P_1 + P_2$, $3P_2$ or $2P_3$;
6. $H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.

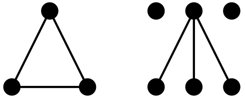
Theorem (D., Paulusma 2015)

This leaves 13 cases where it is unknown if the clique-width of (H_1, H_2)-free graphs is bounded or not (up to some equivalence relation).

1. $H_1 = 3P_1, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$;
3. $H_1 = P_1 + P_4, \overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

There are 15 classes of (H_1, H_2)-free graphs for which both boundedness of clique-width and computational complexity of vertex colouring are open.

$(K_3, K_{1,3} + 2P_1)$-free graphs have bounded clique-width



K_3 $K_{1,3} + 2P_1$

Proof.

- Pick a vertex x. If it has degree < 3, delete it and its neighbourhood. Remainder of the graph is $(K_3, K_{1,3} + P_1)$-free. Clique-width is bounded.
- Let N_1 be the neighbourhood of x. It is an independent set. Let $N_2 = V(G) \setminus (N_1 \cup \{x\})$.
- If N_2 is complete bipartite (or independent), deleting x makes G bipartite, so clique-width is bounded.

□

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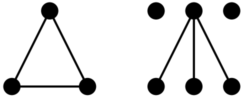

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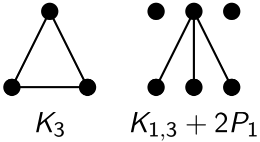

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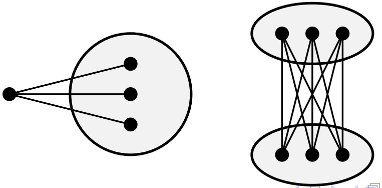
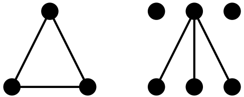
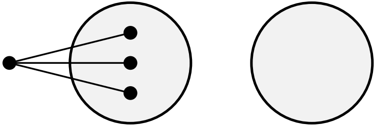


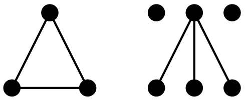


K_3 $\quad$ $K_{1,3} + 2P_1$

- Fix $x_1, x_2, x_3 \in N_1$.



- Given $y_1, y_2, y_3 \in N_2$, at least one y_i must be adjacent to at least one x_j.
- Delete at most two vertices from N_2. Every vertex of N_2 has a neighbour in $\{x_1, x_2, x_3\}$



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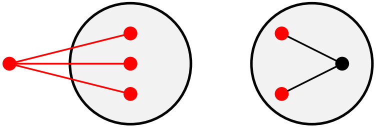

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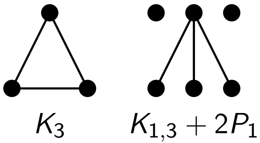

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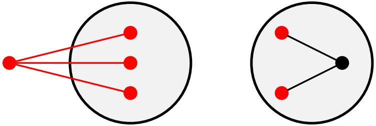

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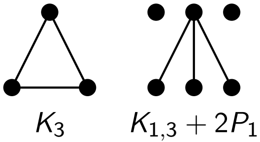
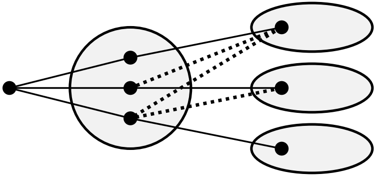
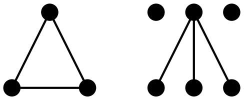


- Partition N_2 as follows: A set of vertices adjacent to x_1, B set of vertices adjacent to x_2, but not x_1, C set of vertices adjacent to x_3, but not x_1 or x_2.



- A, B, C are independent
- If $|C| \geq 3$ get a $K_{1,3} + 2P_1$. May assume C is empty.
- Assume $|A|, |B| \geq 9$, otherwise can delete them and make N_2 independent.
- If $a_1, a_2, a_3 \in A$ and $b_1, b_2 \in B$ then some a_i is adjacent to a b_j
- If $a \in A$ has three neighbours in B then it is complete to B (and vice versa)



K_3 $\quad$ $K_{1,3} + 2P_1$

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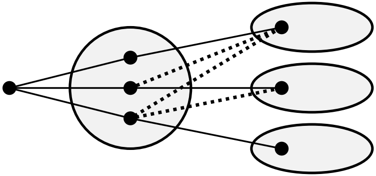

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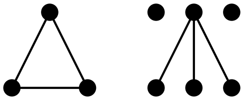

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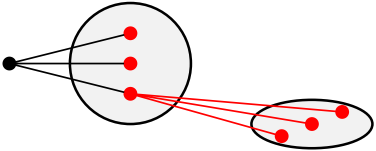

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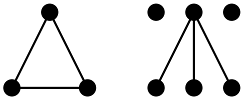

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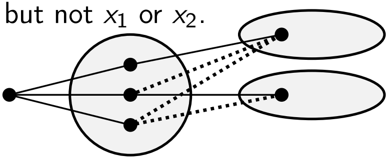

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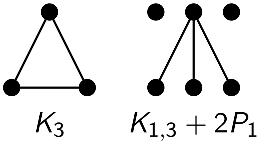

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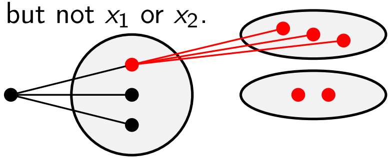

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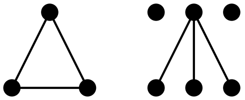

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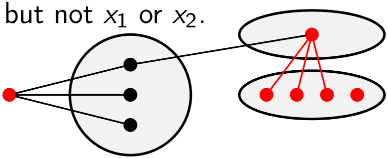

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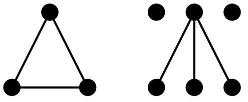

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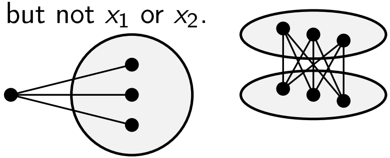

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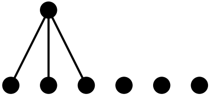
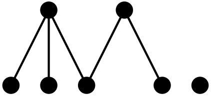
H-free Bipartite Graphs

Theorem (D., Paulusma 2014)

The class of H-free bipartite graphs has bounded clique-width if and only if H is an induced subgraph one of:

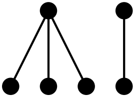


$K_{1,3} + 3P_1$

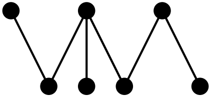


$P_1 + S_{1,1,3}$

sP_1 for some s
($s = 5$ shown)

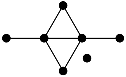


$K_{1,3} + P_2$

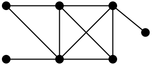


$S_{1,2,3}$

H-free Split Graphs



F_4

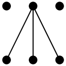


F_5

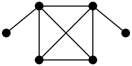
Theorem (Brandstädt, D., Huang, Paulusma, 2015)

Let H be a graph such that neither H nor $\overline{H}$ is in $\{F_4, F_5\}$. The class of *H-free split* *graphs has bounded clique-width if and only if H or $\overline{H}$ is*

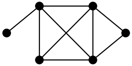
- *isomorphic to rP_1 for some $r \geq 1$ or*
- *an induced subgraph of one of:*



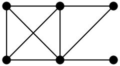
$K_{1,3} + 2P_1$



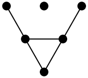
F_1



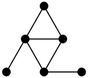
F_2



F_3



bull $+ P_1$



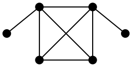
Q

H-free Weakly Chordal Graphs

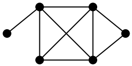
Theorem (Brandstädt, D., Huang, Paulusma 2015)

Let H be a graph. Then the class of H-free weakly chordal graphs has bounded clique-width if and only if $H \subseteq_i P_4$.

H-free Chordal Graphs



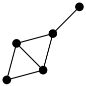
F_1



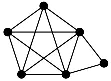
F_2

Theorem (Brandstädt, D., Huang, Paulusma 2015)

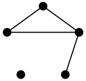
Let H be a graph with $H \notin \{F_1, F_2\}$. The class of H-free chordal graphs has bounded clique-width if and only if H is a an induced subgraph of one of:



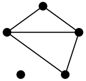
$\overline{S_{1,1,2}}$



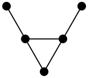
$\overline{K_{1,3} + 2P_1}$



$P_1 + \overline{P_1 + P_3}$



$P_1 + \overline{2P_1 + P_2}$



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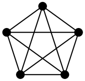
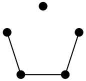
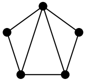


K_r for $r = 5$



$P_1 + P_4$



$\overline{P_1 + P_4}$

Other Containment Relations

Theorem (D., Paulusma 2015)

Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. Then the following statements hold:

(i) *The class of $(H_1, \ldots, H_p)$-subgraph-free graphs has bounded clique-width if and only if $H_i \in \mathcal{S}$ for some $1 \leq i \leq p$.*

(ii) *The class of $(H_1, \ldots, H_p)$-minor-free graphs has bounded clique-width if and only if H_i is planar for some $1 \leq i \leq p$.*

(iii) *The class of $(H_1, \ldots, H_p)$-topological-minor-free graphs has bounded clique-width if and only if H_i is planar and has maximum degree at most 3 for some $1 \leq i \leq p$.*

Summary of Open Problems

For which pairs of graphs (H_1, H_2) does the class of (H_1, H_2)-free graphs have bounded clique-width? (13 open cases: see also "Clique-width of Graph Classes Defined by Two Forbidden Induced Subgraphs" D. & Paulusma, CIAC 2015 and arXiv:1405.7092.)

For which graphs H does the class of H-free chordal graphs have bounded clique-width? (2 open cases: see also "Bounding the Clique-Width of H-free Chordal Graphs" Brandstädt, D., Huang, Paulusma, MFCS 2015 and arXiv:1502.06948.)

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Thank You!